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Abstract: A simple fluid model for the joint throughput evolution of a set of TCP sessions sharing a common bottleneck router has been introduced in [2] based on products of random matrices. This paper studies the asymptotics of the aimd model when the number of TCP sessions N goes to infinity. We show that this limit process can be defined by a recurrence relation in $\mathbb{R}$ and we show how to characterize its stationary behavior very simply based on the recurrence relation. We study the accuracy of this limit model and the impact of the synchronization effect not only on the mean throughput but also on its QoS.

Key-words: TCP/IP, additive increase multiplicative decrease, QoS, synchronization, asymtotic analysis, recurrence relation

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Analyse asymptotique du modèle AIMD pour plusieurs connections TCP

Résumé : Un modèle fluide simple pour l'évolution jointe des débits d'un ensemble de sources TCP partageant un routeur commun a été introduit dans [2] basé sur des produits de matrices aléatoires. Cet article étudie les asymptotiques du modèle AIMD lorsque le nombre de sources TCP N tend vers l'infini. Nous montrons que ce processus limite peut être défini par une relation de récurrence dans $\mathbb{R}$ et nous caractérisons son comportement stationnaire très simplement à partir de cette récurrence. Nous étudions la précision de ce modèle limite et l'impact de la synchronisation non seulement sur le débit moyen mais aussi sur la QoS.

Mots-clés : TCP/IP, accroissements additifs et décroissance multiplicative, QoS, synchronisation, analyse asymptotique, relation de récurrence.

1 Introduction

In [2], a simple model for the joint evolution of TCP controlled N FTP sessions sharing a single bottleneck router was given based on products of random matrices. In this model (as in [2] we will call it the *AIMD* - Additive Increase, Multiplicative Decrease - model), TCP is not represented at packet level, but via simple fluid equations that describe the joint evolutions of the windows in the congestion avoidance phase. The assumptions of the AIMD model were motivated by the small time scale analysis and in particular by the fact that even in this simplified model fractal behavior of aggregated traffic was already present.

The present paper studies the asymptotic AIMD model when the number of sessions N goes to infinity. The main advantage of such a consideration is that this limit process is very easily defined by a one dimensional recurrence relation, even when the loss probability is rate dependent. This enables us to study in particular how the stationary law of the window depends on the synchronization rate.

In §2, we recall briefly the AIMD model. In §3, we present limit theorems when the loss probability is rate independent and in §4 we consider the rate dependent case.

2 The AIMD Model

The AIMD model is a fluid approximation of window evolutions in congestion avoidance phase for N FTP sessions. We recall that slow start and timeout are not taken account in this model. However, timeout is indirectly present by the fact that the throughput can be arbitraly close to 0 (cf. [2]). Besides RTT is assumed to be constant in time and the same for all sessions (homogeneity). If the window size of session i at time t is $W^{i,N}(t)$, its instantaneous throughput or local throughput at time t is approximated by the quantity $X^{i,N}(t) = W^{i,N}(t)/RTT$.

The n-th *congestion time* is defined as the n-th epoch at which one loss or several simultaneous losses may occur on the shared router. We use the following notation:

- N is the number of parallel FTP sessions, which we assume to be constant in time;
- $C(N) = cN$ is the capacity of the bottleneck router;
- $RTT = R$ (constant in time and equal for all sessions);
- T_n is the n-th congestion time;
- $\tau_{n+1} = T_{n+1} - T_n$ is the time between the n-th and the $(n+1)$-st congestion times;
- $X_n^{i,N} = X^{i,N}(T_n^+)$ is the throughput of session i just after the n-th congestion time;
- $S_n^N = \sum_{i=1}^N X_n^{i,N}$;
- $Y_n^{i,N} = X^{i,N}(T_n^-)$ is the throughput of session i just before the n-th congestion time;
- *synchronization probability*: $\gamma_n^i[y]$ is $\{1/2, 1\}$-valued random variable. It is equal to $1/2$ is the session i experiences a loss at T_n. Its law is assumed given and depend a priori on y which is the throughput of the session i just before T_n.
  We set $f(y) = \mathbb{P}(\gamma_n^i[y] = 1/2)$. If $X_n^{i,N}$ is stationary, the stationary probability that $\gamma_n^i[\cdot] = 1/2$ defines the *synchronization rate* that we denote by $\mathbb{E}[f]$.

Here we allow that $\tau_n = 0$. This implies that we may have $X_{n+1}^{i,N} = X_n^{i,N}$, but what is interesting is the property that given y, $\gamma_n^i[y]$ does not depend on i. If we impose $\tau_n > 0$, given y, $\gamma_n^i[y]$ does not depend on i conditional on the fact that there is at least one loss. Allowing $\tau_n = 0$ simplifies moments calculation and has no impact on the asymptotic analysis: indeed, we will see that in the limit model $\tau_n > 0$ with probability 1.

Then, we have

$$X_{n+1}^{i,N} = \gamma_n^i[Y_n^{i,N}] \times Y_n^{i,N} \tag{1}$$

with

$$Y_n^{i,N} = X_n^{i,N} + c - \frac{S_n^N}{N}.$$

For simplicity, we assume that the initial distribution of $X_0^{i,N} \in [0,c]$ is given and taken independently for all i, such that $\mathbb{E}[X_0^{i,N}]$ does not depend on (i,N). As we will see this is not restrictive since for all N, $X_n^{i,N}$ is ergodic.

3 Independent losses

3.1 Asymptotic model characterization

Here we assume that $\gamma_n^i[y]$ is independent of y and that $\mathbb{P}(\gamma_n^i = 1/2) = p = \mathbb{E}[f]$.

Theorem 1. *We have the following L_1 and a.s. limit:*

$$\lim_{N\to\infty} \frac{S_n^N}{N} = \mathbb{E}[X_n^{i,\infty}],$$

where $X_n^{i,\infty}$ is an a.s. limit of $X_n^{i,N}$.

The proof of the above theorem is given in the next section in a more general context. The above theorem implies that $\lim_{n\to\infty}\lim_{N\to\infty}\frac{S_n^N}{N} = (1-p/2)c$.

Theorem 2. $\forall K \in \mathbb{N}$, $\forall (l_1,\cdots,l_K) \in \mathbb{N}^K$,

$$\mathbb{E}\left[\prod_{k=1}^K (X_n^{i_k,N})^{l_k}\right] - \prod_{k=1}^K \mathbb{E}[(X_n^{i_k,N})^{l_k}] = O(1/N), \tag{2}$$

where $i_1,\cdots,i_K$ are K different elements of $\mathbb{N}$.

Proof. The proof is by induction on the degree $l = \sum_{k=1}^K l_k$. For $l = 2$, we have:

$$\begin{aligned}
\mathbb{E}[X_{n+1}^{1,N}X_{n+1}^{2,N}] &= \mathbb{E}[\gamma_{n+1}^1\gamma_{n+1}^2]\left(\mathbb{E}[X_n^{1,N}X_n^{2,N}] + c^2 + \mathbb{E}\left[\frac{(S_n^N)^2}{N^2}\right] - 2\mathbb{E}[X_n^{2,N}\frac{S_n^N}{N}]\right) \\
&= (\mathbb{E}[\gamma])^2\left(c^2 + \frac{1}{N}(\mathbb{E}[X_n^{1,N}X_n^{2,N}] - \mathbb{E}[(X_n^{1,N})^2])\right) \\
&= (\mathbb{E}[(X_n^{1,N})])^2 + O(N^{-1})
\end{aligned}$$

Now for a given $(l_1, \cdots, l_K)$:

$$\begin{aligned} A = \mathbb{E}\left[\prod_{k=1}^{K}(X_{n+1}^{k,N})^{l_k}\right] &= \alpha\mathbb{E}\left[\prod_{k=1}^{K}\left((X_n^{k,N}+c-\frac{S_n^N}{N})^{l_k}\right)\right] \\ &= \alpha\mathbb{E}\left[\prod_{k=1}^{K}\left(\sum_{j_k=0}^{l_k}\binom{l_k}{j_k}(X_n^{k,N}-\frac{S_n^N}{N})^{j_k}c^{l_k-j_k}\right)\right], \end{aligned}$$

where $\alpha = \prod_{k=1}^{K}\mathbb{E}[(\gamma_1^k)^{l_k}]$. For the product of the moments, the expressions follow in the same way

$$\begin{aligned} B = \prod_{k=1}^{K}\mathbb{E}\left[(X_n^{k,N})^{l_k}\right] &= \alpha\prod_{k=1}^{K}\mathbb{E}\left[\left(X_n^{k,N}+c-\frac{S_n^N}{N}\right)^{l_k}\right] \\ &= \alpha\prod_{k=1}^{K}\left(\sum_{j_k=0}^{l_k}\binom{j_k}{l_k}\mathbb{E}[(X_n^{k,N}-\frac{S_n^N}{N})^{j_k}]c^{l_k-j_k}\right). \end{aligned}$$

Now one can easily prove the relation:

$$\mathbb{E}[(X_n^{1,N})^{q_1}\dots(X_n^{k,N})^{q_L}\left(\frac{S_n^N}{N}\right)^m] = \mathbb{E}[(X_n^{1,N})^{q_1}\dots(X_n^{k,N})^{q_L}X_n^{k+1,N}\dots X_n^{k+m,N}] + O(N^{-1}),$$

for all $q_i, m, k \in \mathbb{N}$. Since the coefficients of the expansion of A and B are all equal, $A - B$ gives terms of the form:

$$\mathbb{E}\left(\prod_{k=1}^{K'}(X_n^{k,N})^{n_k}\right) - \prod_{k=1}^{K'}\mathbb{E}\left((X_n^{k,N})^{n_k}\right),$$

with $K' \geq K$ and $n_k \leq l_k$ if $k \leq K$, $\sum_k n_k \leq l$. For the terms such that $\sum_k n_k < l$, we use induction assumption to bound it by $O(1/N)$. The term such that $\sum_k n_k = l$ and $K' = K$ is equal to $\alpha(A - B)$ with $\alpha < 1$, hence this term can be moved to the left hand side. For the terms such that $\sum_k n_k = l$ and $K' > K$, we reapply the above expansion method. In a finite number of steps, we find on the right hans side only terms of the form

$$\mathbb{E}\left(\prod_{k=1}^{l}(X_n^{k,N})^{n'_k}\right) - \prod_{k=1}^{l}\mathbb{E}\left((X_n^{k,N})^{n'_k}\right)$$

with $\sum_k n'_k < l$. □

Corollary 1. *If $\mathbb{S}$ and $\mathbb{S}'$ are two fixed disjoint subsets of $\{X_n^{i,N}\}_{i=1..N}$, then they are asymptotically independent.*

Corollary 2. *The stationnary limit process is characterized by:*

$$X_{n+1}^{\infty} = \gamma_{n+1}(X_n^{\infty} + \frac{pc}{2}) \tag{3}$$

or equivalently by:

$$X_{\infty}^{\infty} = \frac{pc}{2}\sum_{n=1}^{\infty}\prod_{i=1}^{n}\gamma_{-i}$$

Using the recurrence relation (1) and (3), on can easily get all moments of X_N^∞ and X_∞^∞ by induction. Here are the 3 first moments:

$$
\begin{aligned}
\mathbb{E}[X_\infty^\infty] &= \mathbb{E}[X_\infty^{i,N}] = (1-\frac{p}{2})c \\
\mathbb{E}[(X_\infty^{i,\infty})^2] &= \frac{4}{3}(1-\frac{3p}{4})(1-\frac{p}{4})c^2 \\
\mathbb{E}[(X_\infty^{i,N})^2] &= \mathbb{E}[(X_n^{i,\infty})^2](1+\frac{1-p}{3N})^{-1} \\
\mathbb{E}[(X_\infty^{i,\infty})^3] &= \frac{c^3}{8}(\frac{8}{7}-p)(2-p)(8-p) \\
\mathbb{E}[(X_\infty^{i,N})^3] &= \frac{c^3}{8}\frac{3(8-7p)(2-p)(8-p)+\frac{4}{N}(1-p)(8-7p)(6-6p+p^2)}{21+\frac{34}{N}(1-p)+\frac{3}{N^2}(5-7p)(1-p)+\frac{2}{N^3}(1-2p)(1-p)^2}.
\end{aligned}
$$

3.2 Stationary distribution plots

Figure 1 shows the relative difference between $X_\infty^{i,\infty}$ and $X_\infty^{i,N}$ with $N = 10, 100$ for the 2nd and 3rd moments using the analytic formulas given above and simulation results. We see that for the 2nd and 3rd moments, the limit $N = \infty$ gives upper bound estimation for all values of p. The difference decreases almost proportionally to $1-p$ and $1/N$ and $p = 0$ gives a bound on the relative difference: for the 2nd moment, the bound is $\frac{1}{3N} + o(1/N)$, and for the 3rd moment, the bound is $\frac{1.12}{N} + o(1/N)$.

The comparison with simulation shows that 10^7 iterations give a quite good approximation. This gives an idea of the number of iterations that is needed to have relevant results from simulations in the rate dependent synchronization case.

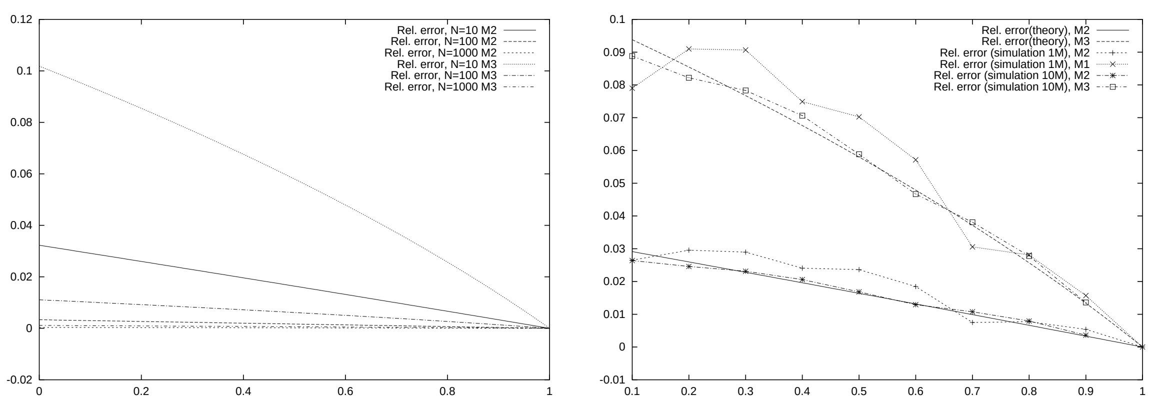


Figure 1: Theoretical relative error plot(left) & Comparison with simulation (right, 10^6 and 10^7 iterations)

Figure 2 shows the empirical distribution (ED) plot and the cumulative distribution function (CDF) plot of the continuous time stationary state of the throughput that we approximate by 10^6 iterations of (3) with $c = 10$ and granularity on $x = 0.01$ for $p \in \{0.1, 0.3, 0.5, 0.7, 0.9, 1\}$. It roughly shows three typical shapes of the ED functions depending on p: $p \le 0.5$, $0.5 \le p < 1$ and $p = 1$.

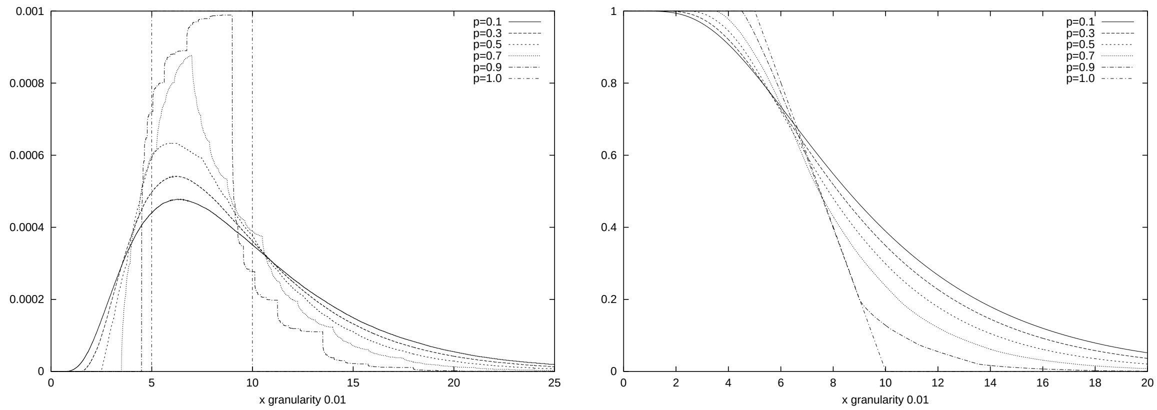


Figure 2: ED and CDF plot

Figure 3 shows the QoS curve plot as a function of p. The definition of QoS $\alpha\%$ is defined as the maximum instantaneous throughput (for the continuous time process) that one gets $\alpha\%$ of time:

$$\text{QoS } \alpha\% = \sup_{x \in \mathbb{R}^+} \{x : \mathbb{P}(X(t) > x) \geq \alpha/100\}$$

(cf. [3] for more details on this definition). It is surprising that when the synchronization rate increases, the QoS $\alpha\%$ curves for hight value of α is increasing whereas the mean throughput is decreasing. This can be partially explained by the fact that the variance of the throughput is a decreasing function of p and this has a major impact on higher QoS. In such a context, the synchronization has a positive effect on the high QoS! One can also note that the region $\alpha \sim 0.7$ is almost insensitive to the synchronization effect.

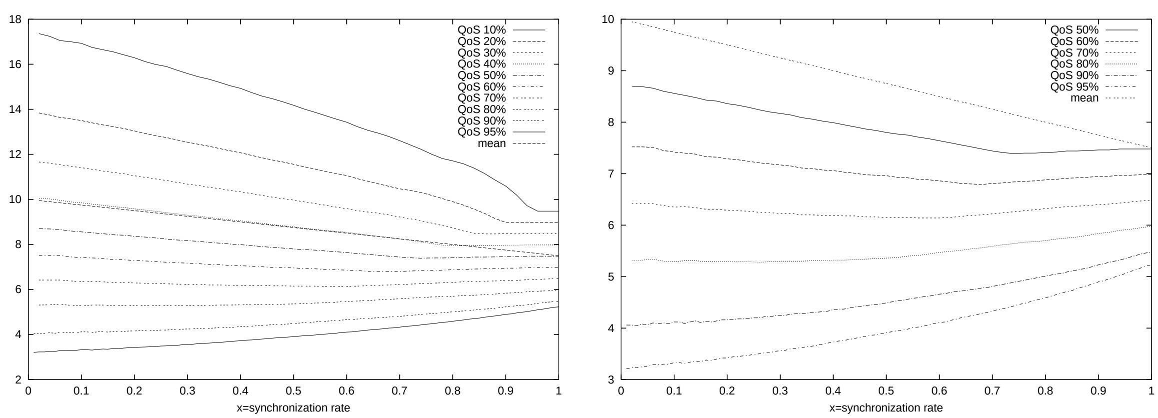


Figure 3: QoS plot

In Figure 3, we also see that for all values of p, the mean throughput is between QoS 40% and QoS 50%. We already know that it cannot be better than the deterministic case $p = 1$ which corresponds to QoS 50%. For $p < 0.8$, the mean throughput is guaranteed 40% of time.

4 Dependent losses

4.1 Asymptotic model characterization

In this section, we give results based on classical analysis for Markov processes on general state space: the basic notions on irreducibility and Harris recurrence can be found in (cf. [6]).

Theorem 3. $\mathbf{X}_n^N = (X_n^{1,N}, \cdots, X_n^{N,N})$ *is a Markov process.*
If $\inf_{y\in\mathbb{R}^+} f(y) = \underline{f} > 0$ *and* $\sup_{y\in\mathbb{R}^+} f(y) = \overline{f} < 1$*, then* $\forall N \in \mathbb{N}$*,* $\mathcal{X}^N = \{\mathbf{X}_n^N\}_{n\in\mathbb{N}}$ *is positive Harris and therefore it is ergodic.*

Proof. The Markov property is clear. $\mathcal{X}^N$ is ϕ-irreducible (cf. [6], p.87), where ϕ is for instance the Lebesgue measure on $\mathbb{R}^+$. To prove this, it is sufficient to show that from any state X, $\mathbf{X}_n^N$ comes back to any given open interval containing $(c/2, \cdots, c/2)$ with positive probability: if all sources experience a loss at T_n, then $S_n^N/N = c/2$, and

$$|X_{n+1}^{i,N} - c/2| = \frac{1}{2}|X_n^{i,N} - c/2|.$$

This event occurs with probability larger than $\underline{f}^N$ and can be repeated any finite number of times with positive probability.

Finally, $\mathcal{X}^N$ is positive Harris by application of Foster's criterion on $\mathbf{X}_n^N$ (cf. [6], p.262). □

Theorem 4. *If* $\inf_{y\in\mathbb{R}^+} f(y) = \underline{f} > 0$*,* $\sup_{y\in\mathbb{R}^+} f(y) = \overline{f} < 1$ *and if* f *is continuous non-decreasing, then:*

$$\lim_{N\to\infty} \frac{S_n^N}{N} = s_n, \qquad \textit{in } L_1 \textit{ and a.s.}$$

where $s_n = \mathbb{E}[X_n^\infty] = \lim_{N\to\infty} \mathbb{E}[X_n^{i,N}]$.
$\forall n \in \mathbb{N}$, $X_n^{i,N} \to X_n^{i,\infty}$ *a.s. and this limit process satisfies the recurrence relation:*

$$X_{n+1}^{i,\infty} = \gamma_{n+1}^i[X_n^{i,\infty} + c - s_n](X_n^{i,\infty} + c - s_n).$$

$\{X_n^{i,\infty}\}_{i>0}$ *are i.i.d. and* $\mathcal{X}^\infty = \{X_n^\infty\}_{n\in\mathbb{N}}$ *is stationary ergodic, hence its stationary law is characterized by the relation*

$$X_{n+1}^\infty = \gamma_{n+1}[X_n^\infty + a](X_n^\infty + a),$$

where $a = c - \mathbb{E}[X_\infty^\infty]$.

Proof. We first show limits of $X_n^{i,N}$ and $\frac{S_n^N}{N}$ by induction. For $n = 0$, $X_0^{i,N} = X_0^{i,\infty}$ by construction and by assumption $\{X_0^{i,N}\}_{i=1,..,N}$ is i.i.d., hence $\frac{S_0^N}{N} \to \mathbb{E}[X_0^{i,\infty}]$ a.s. by law of large numbers. Assume that $X_n^{i,N} \to X_n^{i,\infty}$ and $\frac{\sum_{i=1}^N |X_n^{i,\infty} - X_n^{i,N}|}{N} \to 0$ a.s. We have

$$X_{n+1}^{i,N} = \gamma_{n+1}^i[Y_n^{i,N}](Y_n^{i,N}).$$

Since $Y_n^{i,N} \to X_n^{i,\infty} + c - s_n$ a.s. we have $X_{n+1}^{i,N} \to X_{n+1}^{i,\infty}$ a.s. (the a.s. convergence for $\gamma_{n+1}^i[Y_n^{i,N}]$ is due to the continuity of f).
For the a.s. limit of $\frac{S_{n+1}^N}{N}$, we use the inequality:

$$\begin{aligned} |X_{n+1}^{i,\infty} - X_{n+1}^{i,N}| &\leq \gamma_{n+1}^i[Y_n^{i,\infty}]\left(|Y_n^{i,\infty} - Y_n^{i,N}|\right) + |\gamma_{n+1}^i[Y_n^{i,\infty}] - \gamma_{n+1}^i[Y_n^{i,N}]|Y_n^{i,N} \\ &\leq |Y_n^{i,\infty} - Y_n^{i,N}| + |\gamma_{n+1}^i[Y_n^{i,\infty}] - \gamma_{n+1}^i[Y_n^{i,N}]|Y_n^{i,N} \end{aligned}$$

Therefore

$$\frac{\sum_{i=1}^N |X_{n+1}^{i,\infty} - X_{n+1}^{i,N}|}{N} \leq \frac{\sum_{i=1}^N |X_n^{i,\infty} - X_n^{i,N}|}{N} + \left|\frac{S_n^N}{N} - s_n\right| + \sum_{i=1}^N \frac{\mathbf{1}_{\gamma_{n+1}^i[Y_n^{i,\infty}] \neq \gamma_{n+1}^i[Y_n^{i,N}]}}{N}(n+1)c.$$

Now we show by weak induction that $\frac{\sum_{i=1}^N \mathbf{1}_{\gamma_{n+1}^i[Y_n^{i,\infty}] \neq \gamma_{n+1}^i[Y_n^{i,N}]}}{N} \to 0$ a.s. For the initial condition: $s_0 - \frac{S_0^N}{N} \to 0$ a.s., hence almost surely we have for all $\varepsilon > 0$, for N large enough (s.t. $|S_0^N/N - s_0| \leq \varepsilon$):

$$\mathbf{1}_{\gamma_1^i[Y_0^{i,\infty}] \neq \gamma_1^i[Y_0^{i,N}]} \leq \mathbf{1}_{\gamma_1^i[Y_0^{i,\infty}] \neq \gamma_1^i[Y_0^{i,\infty}+\varepsilon]} + \mathbf{1}_{\gamma_1^i[Y_0^{i,\infty}] \neq \gamma_1^i[Y_0^{i,\infty}-\varepsilon]}$$

Using the law of large numbers,

$$\frac{\sum_{i=1}^N \mathbf{1}_{\gamma_1^i[Y_0^{i,\infty}] \neq \gamma_1^i[Y_0^{i,N}]}}{N} \leq f(Y_0^{i,\infty} + \varepsilon) - f(Y_0^{i,\infty} - \varepsilon) \text{ a.s.}$$

Then one can conclude using the fact that f is continuous and $Y_0^{i,\infty}$ bounded.

Now we assume that $\frac{\sum_{i=1}^N \mathbf{1}_{\gamma_{m+1}^i[Y_m^{i,\infty}] \neq \gamma_{m+1}^i[Y_m^{i,N}]}}{N} \to 0$ a.s. for $1 \leq m \leq n-1$. Then

$$\frac{\sum_{i=1}^N \mathbf{1}_{\gamma_{n+1}^i[Y_n^{i,\infty}] \neq \gamma_{n+1}^i[Y_n^{i,N}]}}{N} \leq$$
$$\frac{\sum_{i=1}^N \mathbf{1}_{\gamma_{n+1}^i[Y_n^{i,\infty}] \neq \gamma_{n+1}^i[Y_n^{i,N}]} \prod_{j=0}^{n-1} \mathbf{1}_{\gamma_{j+1}^i[Y_j^{i,\infty}] = \gamma_{j+1}^i[Y_j^{i,N}]}}{N} + \frac{\sum_{i=1}^N \sum_{j=0}^{n-1} \mathbf{1}_{\gamma_{j+1}^i[Y_j^{i,\infty}] \neq \gamma_{j+1}^i[Y_j^{i,N}]}}{N}.$$

We have $\frac{\sum_{i=1}^N \sum_{j=0}^{n-1} \mathbf{1}_{\gamma_{j+1}^i[Y_j^{i,\infty}] \neq \gamma_{j+1}^i[Y_j^{i,N}]}}{N} \to 0$ by assumption and as for the initial condition if for $0 \leq j \leq n$, $|s_j - \frac{S_j^N}{N}| \leq \varepsilon$:

$$\frac{\sum_{i=1}^N \mathbf{1}_{\gamma_{n+1}^i[Y_n^{i,\infty}] \neq \gamma_{n+1}^i[Y_n^{i,N}]} \prod_{j=0}^{n-1} \mathbf{1}_{\gamma_{j+1}^i[Y_j^{i,\infty}] = \gamma_{j+1}^i[Y_j^{i,N}]}}{N} \leq$$
$$\frac{\sum_{i=1}^N \mathbf{1}_{\gamma_{n+1}^i[Y_n^{i,\infty}] \neq \gamma_{n+1}^i[Y_n^{i,\infty}+(n+1)\varepsilon]}}{N} + \frac{\sum_{i=1}^N \mathbf{1}_{\gamma_{n+1}^i[Y_n^{i,\infty}] \neq \gamma_{n+1}^i[Y_n^{i,\infty}-(n+1)\varepsilon]}}{N}.$$

We can then conclude that $\frac{\sum_{i=1}^N |X_{n+1}^{i,\infty} - X_{n+1}^{i,N}|}{N} \to 0$ a.s.

The ergodicity of the limit process is as in Theorem 3: the difficulty here is the irreducibility. Define $\{a_n\}_{n\in\mathbb{N}}$ by the recursion:

$$a_{n+1} = \frac{a_n + s_n}{2}, \quad a_0 \in \mathbb{R}. \tag{4}$$

Let V be the adherent point set of a. First it is clear that V does not depend on the choice of a_0 nor on the changes of a finite number of s_n. Therefore, if $v \in V$, from any point $X_{n_0}^{i,\infty}$, iterating (4) with initial condition $a_{n_0} = X_{n_0}^{i,\infty}$, a_n will in a finite number of steps fall in any open set containing v. Hence, $\{X_n^{i,\infty}\}_{n\in\mathbb{N}}$ is irreducible. And $s_n = \mathbb{E}[X_n^{i,\infty}]$ converges to $s_\infty = \mathbb{E}[X_\infty^{i,\infty}]$, where $X_\infty^{i,\infty}$ is the stationary state of $X_n^{i,\infty}$. □

Comparison with the $1/\sqrt{p}$ formula Let p_l be the stationary probability that one packet is lost. This can be expressed as the ratio between the number of packets lost and the number of packets sent:

$$p_l = \frac{\mathbb{E}[f]}{\mathbb{E}[\tau]\mathbb{E}[X(t)]}.$$

Based on the approximation (6) below, we have the relation:

$$\text{Throughput} \sim \frac{\sqrt{2(1-\mathbb{E}[f]/4)}}{RTT\sqrt{p_l}}. \tag{5}$$

As we will see by simulations, the formula (6) is very robust, thus the coefficient $\sqrt{2(1-\mathbb{E}[f]/4)}$ gives a good idea of how, given RTT and p_l, the throughput depends also on the synchronization rate, even if one does not know the exact synchronization probability function. The relation (5) is an equality if the synchronization probability does not depend on the rate.

4.2 Stationary distribution plots

We call MOD1 the case with rate independent synchronization probability. In the following, the value of c has been fixed to 10.

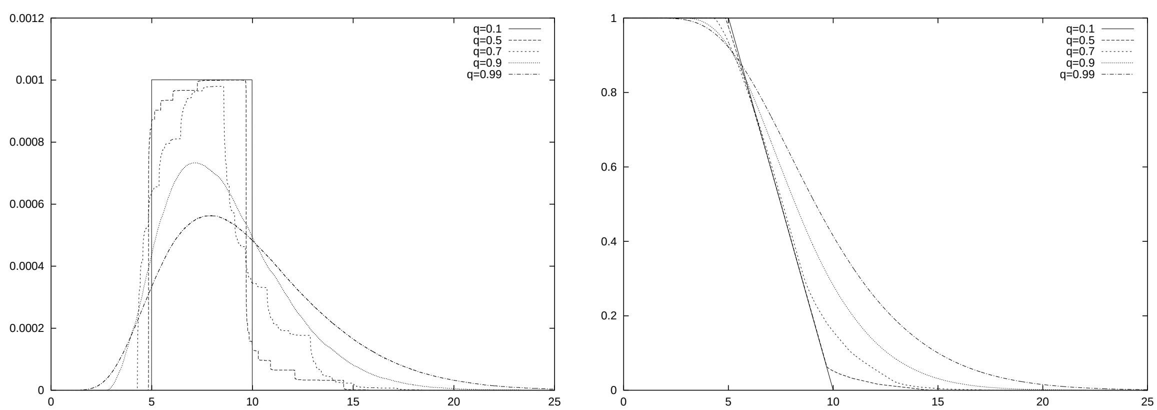


Figure 4: ED and CDF plot

Figure 4 shows the ED plot and the CDF plot of $X(t)$ when the synchronization probability is given by $p = 1 - q^x$ with $q = 0.1, 0.5, 0.7, 0.9, 0.99$ (case MOD2). In Figure 5, we chose $p = MIN(x/K, 0.9)$ with $K = 5, 6, 10, 100$ (case MOD3). We see that we find again the 3 typical curve shapes of case MOD1.

Figure 6 shows the relative difference between $X_\infty^{i,\infty}$ and $X_\infty^{i,N}$ with $N = 10$ for the 3 first moments as a function of the synchronization rate. Plots are obtained by simulation of 10^7 iterations of the recurrence relations. We see that for the 2nd and 3rd moments, the limit $N = \infty$ does not always give an upper bound (only for $p < 0.5$). Note that for $N = 10$, differences are less than 4% for the 3rd moment and less than 1% for the 2nd moment, which is better than MOD1 case. Since the relative differences should decrease with N, this means that the limit model offers a very good approximation even if N is not that large, at least for the three synchronization probability models we proposed.

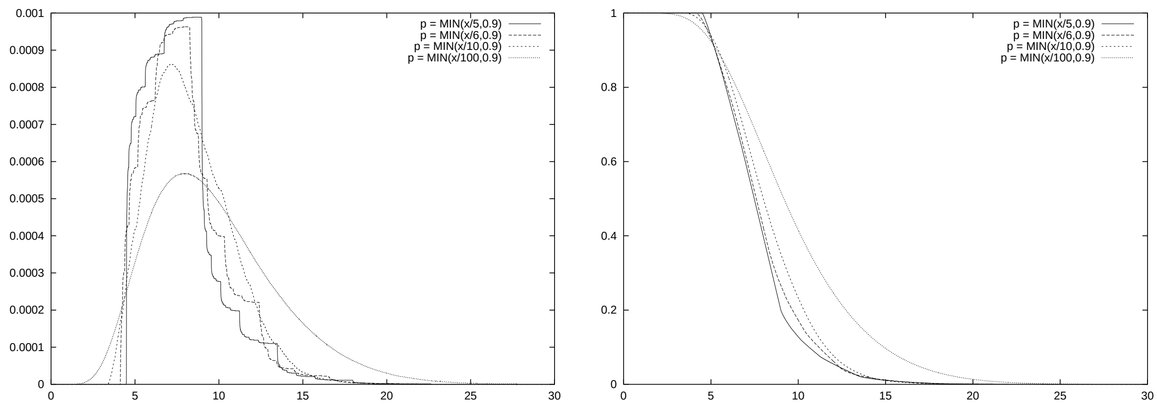


Figure 5: ED and CDF plots.

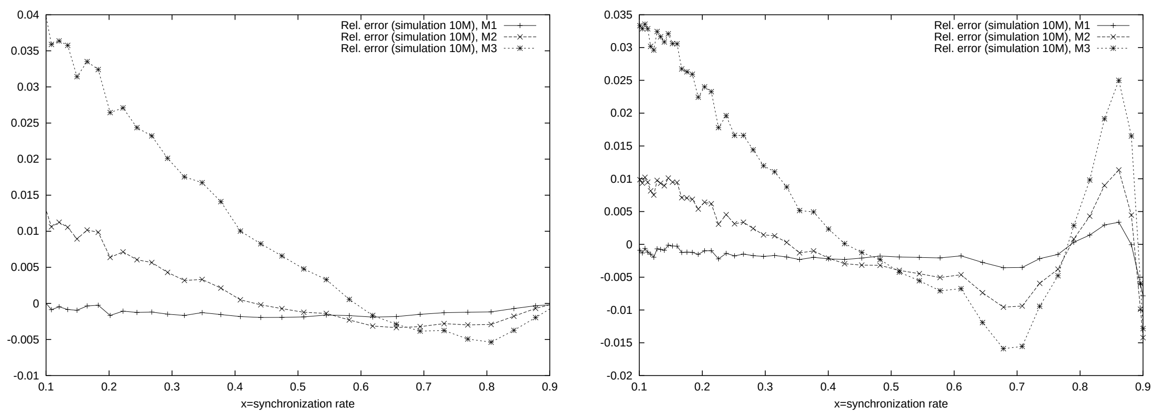


Figure 6: Relative error plot by simulation (MOD2 left, MOD3 right) after 10^7 iterations.

Figure 7 shows comparison of QoS plots between MOD1, MOD2 and MOD3 as a function of the synchronization rate. For high value of α, the QoS $\alpha\%$ are better than MOD1: this can be explained by the fact that rate dependent case tends to reduce the fluctuations of throughput, therefore its variance. Besides, the high QoS is still an increasing function in the synchronization rate, but its slope is much smaller than the case MOD1 and QoS 90% is quite insensitive w.r.t. the synchronization rate.

One can also see that should one know the synchronization rate, the mean throughput is very well approximated using the formula:

$$\mathbb{E}[X_n] \sim (1 - \mathbb{E}[f]/2)c \tag{6}$$

and the relation

$$\mathbb{E}[X(t)] = \mathbb{E}[(X_n + c)/2] \sim (1 - \mathbb{E}[f]/4)c. \tag{7}$$

The mean throughput is guaranteed about 45% of time (for $p < 0.9$, both for MOD2 and MOD3), which is 10% less than the best one can hope (50% for $p = 1$).

Figure 8 shows comparison of distribution plots between MOD1, MOD2 and MOD3 when we choose parameters of synchronization probability such that the synchronization rate $\mathbb{E}[f]$

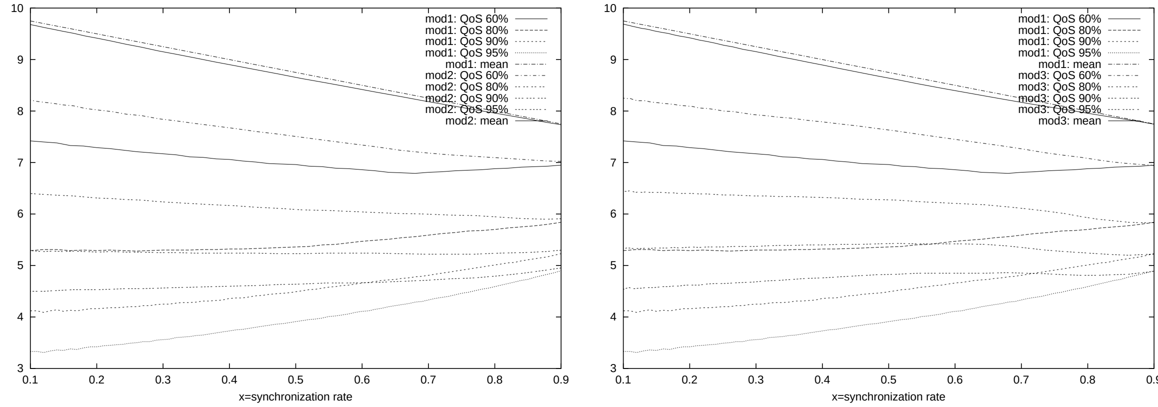


Figure 7: Comparison of QoS

is equal to 0.5 and 0.8. We see that when the synchronization rate increases, the difference between MOD1, MOD2 and MOD3 collapses. With $\mathbb{E}[f] = 0.8$, we see that ED and CDF plots are already very close. For instance, with $\mathbb{E}[f] = 0.5$, the difference of QoS 90% between independent and dependent models is less than 20%, with $\mathbb{E}[f] = 0.8$, it is less than 5%.

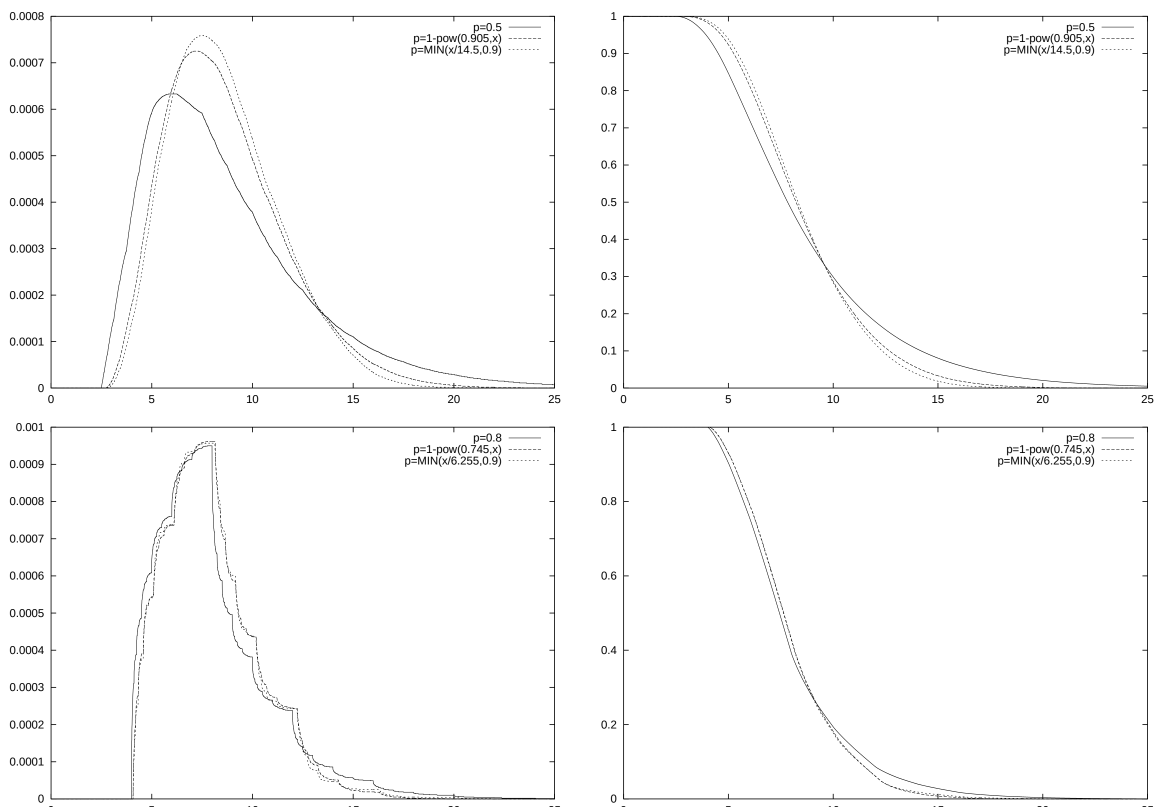
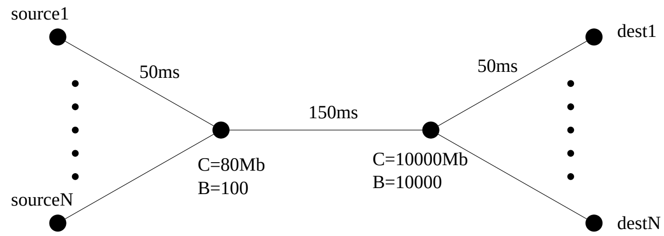


Figure 8: Comparison of distribution functions: Synchronization rate 50%(top), 80%(bottom)

4.3 Comparison with NS

The distribution function we obtained by iterations of the recurrence relation (1) can be compared to that given by NS simulation.



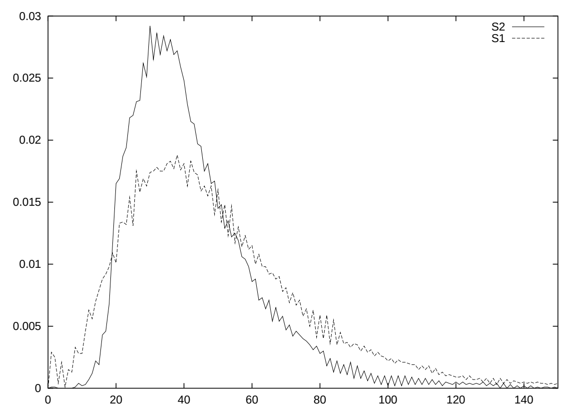


Figure 9: ED of window size by NS simulation

Figure 9 shows the ED plots obtained in the following settings (simulation time = 10000s):

S1: $N = 100$, $C = 80Mb$, $MSS = 1Kbytes$ (10000 pkts/s), Buffer Size $B = 1000$ pkts, $RTT_{\min} = 0.5s$ (pure propagation delay) with *overhead_* set to 0.01.

S2: as S1 with *overhead_* set to 0.001.

It is well known that without *overhead_* (which introduces an additional uniform random delays on ACK send date and plays the role of noises that come from the other routers), NS gives too synchronized window behavior. Table 1 gives some numerical results.

Table 1.

	gput	U	losses(%)	timeouts(%)	$\mathbb{E}$[f]
S1	96.35	0.0365	0.32	6.0 10^{-3}	0.32
S2	85.02	0.1498	0.11	6.0 10^{-5}	0.76

gput: goodput by flow in pkts/s, $U =$ under-utilization factor $= (c - gput)/c$, $\mathbb{E}[f]$ is the synchronization rate estimated from the formula (7) replacing c by $c + \frac{1}{R}(\sqrt{2B/N} + d/R)$ ($\sqrt{2B/N}/R$ is the window increase that fills the buffer cf.[2] and d/R^2 is the window increase during the loss detection delay, here $d \sim R$) which leads to $4U \sim f - \frac{1}{cR}(\sqrt{2B/N} + d/R)(4 - f)$.

Playing with *overhead_*, one can introduce more or less synchronization effect. First we see that in these simulations, timeout probabilities are at least two orders below the loss probabilities and we can reasonably neglect the timeout influence (if not the comparison with the AIMD model is not justified). The two curves we get in Figure 9 match quite well those obtained in MOD1-MOD3 cases with the corresponding value of $\mathbb{E}[f]$. Here the buffer capacity effect cannot be neglected. If B was small, we should have $4U \sim f$. The synchronization rate $\mathbb{E}[f]$ we found by the formula $4U = f - \frac{1}{cR}(\sqrt{2B/N}+1)(4-f)$ leads to a good approximation of what we observed in the NS simulation.

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